

The effective content of Reverse Nonstandard Mathematics

and the nonstandard content of effective Reverse Mathematics

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Abstract. The aim of this paper is to highlight a hitherto unknown *computational* aspect of Nonstandard Analysis pertaining to Reverse Mathematics (RM). In particular, we shall establish RM-equivalences *between theorems from Nonstandard Analysis* in a fragment of Nelson’s *internal set theory*. We then extract *primitive recursive* terms from Gödel’s system T (not involving Nonstandard Analysis) from the proofs of the aforementioned nonstandard equivalences. The resulting terms turn out to be witnesses for *effective*¹ equivalences in Kohlenbach’s *higher-order* RM. In other words, from an RM-equivalence in Nonstandard Analysis, we can extract the associated effective higher-order RM-equivalence *which does not involve Nonstandard Analysis anymore*. Finally, we show that certain effective equivalences in turn give rise to the original nonstandard theorems from which they were derived.

1 Introduction

1.1 Aim of this paper

In two words, this paper deals with a new *computational* aspect of Nonstandard Analysis pertaining to Reverse Mathematics (RM), in line with the results in [7–10]. In particular, we shall prove certain RM-equivalences in (a fragment of) Nelson’s *internal set theory* ([6]), and use the framework from [2] as sketched in Section 1.3, to obtain *effective*¹ equivalences in Kohlenbach’s *higher-order* RM ([4]), where the latter *does not* involve Nonstandard Analysis. Perhaps surprisingly, we also show that from certain effective equivalences, the original nonstandard equivalences can be re-obtained. In other words, there is ‘a two-way street’ between higher-order RM, and the RM of Nonstandard Analysis, as suggested by the title. We refer to [11, 12] for an overview of RM.

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¹ An implication $(\exists \Phi)A(\Phi) \rightarrow (\exists \Psi)B(\Psi)$ is *effective* if there is a term t in the language such that additionally $(\forall \Phi)[A(\Phi) \rightarrow B(t(\Phi))]$, i.e. Ψ can be effectively defined in terms of Φ ; See also Definition 2.1.

1.2 Introducing internal set theory

In Nelson's *syntactic* approach to Nonstandard Analysis ([6]), a new predicate 'st(x)', read as ' x is standard' is added to the language of ZFC, the usual foundations of mathematics. The notations $(\forall^{\text{st}}x)$ and $(\exists^{\text{st}}y)$ are short for $(\forall x)(\text{st}(x) \rightarrow \dots)$ and $(\exists y)(\text{st}(y) \wedge \dots)$. A formula is called *internal* if it does not involve 'st', and *external* otherwise. The three external axioms *Idealisation*, *Standard Part*, and *Transfer* govern the new predicate 'st'; They are respectively defined² as:

- (I) $(\forall^{\text{st}} \text{fin} x)(\exists y)(\forall z \in x)\varphi(z, y) \rightarrow (\exists y)(\forall^{\text{st}} x)\varphi(x, y)$, for internal φ with any (possibly nonstandard) parameters.
- (S) $(\forall x)(\exists^{\text{st}} y)(\forall^{\text{st}} z)((z \in x \wedge \varphi(z)) \leftrightarrow z \in y)$, for internal φ .
- (T) $(\forall^{\text{st}} x)\varphi(x, t) \rightarrow (\forall x)\varphi(x, t)$, where φ is internal, t captures *all* parameters of φ , and t is standard.

Nelson's system IST is ZFC extended with the aforementioned external axioms; IST is a conservative extension of ZFC for the internal language, as proved in [6]. Gödel's system T ([1]) extended with fragments of the external axioms of IST, is studied in [2]. In particular, the latter studies nonstandard extensions of the internal systems E-HA^ω and E-PA^ω , respectively *Heyting and Peano arithmetic in all finite types and the axiom of extensionality*. We refer to [2, §2.1] for the details of the latter (mainstream in mathematical logic) systems.

As to notation, in the aforementioned systems of higher-order arithmetic, each variable x^ρ comes equipped with a superscript denoting its type, which is however often implicit. As to the coding of multiple variables, the type ρ^* is the type of finite sequences of type ρ , a notational device used in [2] and this paper; Underlined variables \underline{x} consist of multiple variables of (possibly) different type.

1.3 A fragment of internal set theory based on Gödel's T

The system P consist of the following axioms, starting with the basic ones.

Definition 1.1 [Basic axioms of P]

1. The system $\text{E-PA}^{\omega*}$ be the definitional extension of E-PA^ω with types for finite sequences as in [2, §2].
2. The set \mathcal{T}^* is the collection of all the constants in the language of $\text{E-PA}^{\omega*}$.
3. The external induction axiom IA^{st} is

$$\Phi(0) \wedge (\forall^{\text{st}} n^0)(\Phi(n) \rightarrow \Phi(n+1)) \rightarrow (\forall^{\text{st}} n^0)\Phi(n). \quad (\text{IA}^{\text{st}})$$

4. The system $\text{E-PA}_{\text{st}}^{\omega*}$ is defined as $\text{E-PA}^{\omega*} + \mathcal{T}_{\text{st}}^* + \text{IA}^{\text{st}}$, where $\mathcal{T}_{\text{st}}^*$ consists of the following axiom schemas.
 - (a) The schema³ $\text{st}(x) \wedge x = y \rightarrow \text{st}(y)$,

² The superscript 'fin' in (I) means that x is finite, i.e. its number of elements are bounded by a natural number.

³ The language of $\text{E-PA}_{\text{st}}^{\omega*}$ contains a symbol st_σ for each finite type σ , but the subscript is always omitted. Hence $\mathcal{T}_{\text{st}}^*$ is an *axiom schema* and not an axiom.

- (b) The schema providing for each closed term $t \in \mathcal{T}^*$ the axiom $\text{st}(t)$.
- (c) The schema $\text{st}(f) \wedge \text{st}(x) \rightarrow \text{st}(f(x))$.

Secondly, Nelson's axiom *Standard part* is weakened in [2] to HAC_{int} :

$$(\forall^{\text{st}} x^\rho)(\exists^{\text{st}} y^\tau)\varphi(x, y) \rightarrow (\exists^{\text{st}} F^{\rho \rightarrow \tau^*})(\forall^{\text{st}} x^\rho)(\exists y^\tau \in F(x))\varphi(x, y), \quad (\text{HAC}_{\text{int}})$$

where φ is any internal formula. Note that F only provides a *finite sequence* of witnesses to $(\exists^{\text{st}} y)$, explaining its name *Herbrandized Axiom of Choice*. Thirdly, Nelson's axiom idealisation I appears in [2] as follows:

$$(\forall^{\text{st}} x^{\sigma^*})(\exists y^\tau)(\forall z^\sigma \in x)\varphi(z, y) \rightarrow (\exists y^\tau)(\forall^{\text{st}} x^\sigma)\varphi(x, y), \quad (\text{I})$$

where φ is any internal formula.

For the full system $\text{P} \equiv \text{E-PA}_{\text{st}}^{\omega*} + \text{HAC}_{\text{int}} + \text{I}$, we have the following theorem. Here, the superscript ' S_{st} ' is the syntactic translation defined in [2, Def. 7.1], and also listed starting with (A.1) in the proof of Corollary 1.3.

Theorem 1.2 *Let $\Phi(\underline{a})$ be a formula in the language of $\text{E-PA}_{\text{st}}^{\omega*}$ and suppose $\Phi(\underline{a})^{S_{\text{st}}} \equiv \forall^{\text{st}} \underline{x} \exists^{\text{st}} \underline{y} \varphi(\underline{x}, \underline{y}, \underline{a})$. If Δ_{int} is a collection of internal formulas and*

$$\text{P} + \Delta_{\text{int}} \vdash \Phi(\underline{a}), \quad (1.1)$$

then one can extract from the proof a sequence of closed terms t in \mathcal{T}^ such that*

$$\text{E-PA}^{\omega*} + \Delta_{\text{int}} \vdash \forall \underline{x} \exists \underline{y} \in t(\underline{x}) \varphi(\underline{x}, \underline{y}, \underline{a}). \quad (1.2)$$

Proof. Immediate by [2, Theorem 7.7].

The following corollary is essential to our results. We shall refer to a formula of the form $(\forall^{\text{st}} \underline{x})(\exists^{\text{st}} \underline{y})\psi(\underline{x}, \underline{y}, \underline{a})$, where φ is internal, as a *normal form*.

Corollary 1.3 *If for internal ψ the formula $\Phi(\underline{a}) \equiv (\forall^{\text{st}} \underline{x})(\exists^{\text{st}} \underline{y})\psi(\underline{x}, \underline{y}, \underline{a})$ satisfies (1.1), then $(\forall \underline{x})(\exists \underline{y} \in t(\underline{x}))\psi(\underline{x}, \underline{y}, \underline{a})$ is proved in the corresponding (1.2).*

Proof. Clearly, if for ψ and Φ as given we have $\Phi(\underline{a})^{S_{\text{st}}} \equiv \Phi(\underline{a})$, then the corollary follows immediately from the theorem. A tedious but straightforward verification using the clauses (i)-(v) in [2, Def. 7.1] establishes that indeed $\Phi(\underline{a})^{S_{\text{st}}} \equiv \Phi(\underline{a})$. For completeness, this verification is performed in Section A.1. \square

Finally, the previous theorems do not really depend on the presence of full Peano arithmetic. Indeed, let E-PRA^ω be the system defined in [4, §2] and let $\text{E-PRA}^{\omega*}$ be its definitional extension with types for finite sequences as in [2, §2]. We permit ourselves a slight abuse of notation by not distinguishing between Kohlenbach's $\text{RCA}_0^\omega \equiv \text{E-PRA}^\omega + \text{QF-AC}^{1,0}$ (See [4, §2]) and $\text{E-PRA}^{\omega*} + \text{QF-AC}^{1,0}$.

Corollary 1.4 *The previous theorem and corollary go through for P and $\text{E-PA}^{\omega*}$ replaced by $\text{RCA}_0^\omega \equiv \text{E-PRA}^{\omega*} + \mathcal{T}_{\text{st}}^* + \text{HAC}_{\text{int}} + \text{I} + \text{QF-AC}^{1,0}$ and RCA_0^ω .*

Proof. The proof of [2, Theorem 7.7] goes through for any fragment of $\text{E-PA}^{\omega*}$ which includes EFA , sometimes also called $\text{I}\Delta_0 + \text{EXP}$. In particular, the exponential function is (all what is) required to 'easily' manipulate finite sequences.

1.4 Notations

We mostly use the same notations as in [2], some of which we repeat here.

Remark 1.5 (Notations) We write $(\forall^{\text{st}} x^\tau) \Phi(x^\tau)$ and $(\exists^{\text{st}} x^\sigma) \Psi(x^\sigma)$ as short for $(\forall x^\tau) [\text{st}(x^\tau) \rightarrow \Phi(x^\tau)]$ and $(\exists x^\sigma) [\text{st}(x^\sigma) \wedge \Psi(x^\sigma)]$. We also write $(\forall x^0 \in \Omega) \Phi(x^0)$ and $(\exists x^0 \in \Omega) \Psi(x^0)$ as short for $(\forall x^0) [\neg \text{st}(x^0) \rightarrow \Phi(x^0)]$ and $(\exists x^0) [\neg \text{st}(x^0) \wedge \Psi(x^0)]$. Furthermore, if $\neg \text{st}(x^0)$ (resp. $\text{st}(x^0)$), we also say that x^0 is ‘infinite’ (resp. finite) and write ‘ $x^0 \in \Omega$ ’. Finally, a formula A is ‘internal’ if it does not involve st , and A^{st} is defined from A by appending ‘ st ’ to all quantifiers (except bounded number quantifiers).

Secondly, we will use the usual notations for rational and real numbers and functions as introduced in [4, p. 288-289] (and [12, I.8.1] for the former).

Definition 1.6 (Real number) A (standard) real number x is a (standard) fast-converging Cauchy sequence $q_{(\cdot)}^1$, i.e. $(\forall n^0, i^0)(|q_n - q_{n+i}| <_0 \frac{1}{2^n})$. We freely make use of Kohlenbach’s ‘hat function’ from [4, p. 289] to guarantee that every sequence f^1 can be viewed as a real. We also use the notation $[x](k) := q_k$ for the k -th approximation of real numbers. Two reals x, y represented by $q_{(\cdot)}$ and $r_{(\cdot)}$ are *equal*, denoted $x =_{\mathbb{R}} y$, if $(\forall n)(|q_n - r_n| \leq \frac{1}{2^n})$. Inequality $<_{\mathbb{R}}$ is defined similarly. We also write $x \approx y$ if $(\forall^{\text{st}} n)(|q_n - r_n| \leq \frac{1}{2^n})$ and $x \gg y$ if $x >_{\mathbb{R}} y \wedge x \not\approx y$. Functions $F : \mathbb{R} \rightarrow \mathbb{R}$ are represented by $\Phi^{1 \rightarrow 1}$ such that

$$(\forall x, y)(x =_{\mathbb{R}} y \rightarrow \Phi(x) =_{\mathbb{R}} \Phi(y)), \quad (\text{RE})$$

i.e. equal reals are mapped to equal reals. Finally, sets are denoted X^1, Y^1, Z^1, \dots and are given by their characteristic functions f_X^1 , i.e. $(\forall x^0)[x \in X \leftrightarrow f_X(x) = 1]$, where f_X^1 is assumed to be binary.

Thirdly, we use the usual extensional notion of equality.

Remark 1.7 (Equality) Equality between natural numbers ‘ $=_0$ ’ is a primitive. Equality ‘ $=_\tau$ ’ for type τ -objects x, y is then defined as follows:

$$[x =_\tau y] \equiv (\forall z_1^{\tau_1} \dots z_k^{\tau_k})[xz_1 \dots z_k =_0 yz_1 \dots z_k] \quad (1.3)$$

if the type τ is composed as $\tau \equiv (\tau_1 \rightarrow \dots \rightarrow \tau_k \rightarrow 0)$. In the spirit of Nonstandard Analysis, we define ‘approximate equality \approx_τ ’ as follows:

$$[x \approx_\tau y] \equiv (\forall^{\text{st}} z_1^{\tau_1} \dots z_k^{\tau_k})[xz_1 \dots z_k =_0 yz_1 \dots z_k] \quad (1.4)$$

with the type τ as above. The system \mathbf{P} includes the *axiom of extensionality*:

$$(\forall x^\rho, y^\rho, \varphi^{\rho \rightarrow \tau})[x =_\rho y \rightarrow \varphi(x) =_\tau \varphi(y)]. \quad (\text{E})$$

However, as noted in [2, p. 1973], the so-called axiom of *standard* extensionality $(\text{E})^{\text{st}}$ is problematic and cannot be included in \mathbf{P} . Finally, a functional $\Xi^{1 \rightarrow 0}$ is called an *extensionality functional* for $\varphi^{1 \rightarrow 1}$ if

$$(\forall k^0, f^1, g^1)[\overline{f} \Xi(f, g, k) =_0 \overline{g} \Xi(f, g, k) \rightarrow \overline{\varphi(f)} k =_0 \overline{\varphi(g)} k], \quad (1.5)$$

i.e. Ξ witnesses (E) for φ . As will become clear in Section 2, $(\text{E})^{\text{st}}$ is translated to the existence of an extensionality functional when applying Corollary 1.3.

2 Main results

In this section, we prove that from certain RM-equivalences in Nonstandard Analysis, one can extract *effective* RM-equivalences in Kohlenbach's higher-order RM. The notion of 'effective' implication is defined as expected.

Definition 2.1 [Effective implication] An implication $(\exists\Phi)A(\Phi) \rightarrow (\exists\Psi)B(\Psi)$ is *effective* if there is a term t in the language such that additionally $(\forall\Phi)[A(\Phi) \rightarrow B(t(\Phi))]$, i.e. Ψ can be effectively defined in terms of Φ .

The terms involved in our effective implications are *primitive recursive* in the sense of Gödel's system T, as discussed in Section 1.3. In light of the elementary nature of an extensionality functional (See Remark 1.7), we still refer to an implication as 'effective', if the term t as in Definition 2.1 involves an extensionality functional. This assumption is not entirely innocent by [5, Remark 3.6].

We shall treat the uniform version of weak König's lemma in Section 2.1 in some detail, by way of illustration. In Section 2.2, we treat the main theorem from [3] pertaining to group theory, using the proofs from Section 2.1 as a template. Similar results for the RM of weak König's lemma may be proved similarly.

2.1 Uniform weak König's lemma

In this section, we first establish a particular nonstandard equivalence involving a fragment of Nelson's axiom *Transfer*, and a (nonstandard and uniform) version of weak König's lemma. As a result of applying Corollary 1.4 to this nonstandard equivalence, we obtain an *effective* equivalence between the uniform version of weak König's lemma UWKL and a version of arithmetical comprehension (μ^2) .

$$(\exists\Phi^{1\rightarrow 1})(\forall T \leq_1 1)[(\forall n)(\exists\alpha)(|\alpha| = n \wedge \alpha \in T) \rightarrow (\forall m)(\overline{\Phi(T)}m \in T)] \quad (\text{UWKL})$$

$$(\exists\mu^2)(\forall f^1)[(\exists x^0)f(x) = 0 \rightarrow f(\mu(f))]$$

The functional (μ^2) is also known as *Feferman's non-constructive mu-operator* (See [1]). We also need the following restriction of Nelson's axiom *Transfer*:

$$(\forall^{\text{st}} f^1)[(\forall^{\text{st}} n^0)f(n) = 0 \rightarrow (\forall m)f(m) = 0]. \quad (\Pi_1^0\text{-TRANS})$$

Define $\text{UWKL}(\Phi, T)$ as UWKL without the leading quantifiers, and UWKL^+ as:

$$(\exists^{\text{st}}\Phi^{1\rightarrow 1})[(\forall^{\text{st}}T^1)\text{UWKL}(\Phi, T) \wedge (\forall^{\text{st}}T^1, S^1)(T \approx_1 S \rightarrow \Phi(T) \approx_1 \Phi(S))],$$

Note that the second conjunct expresses that Φ is *standard extensional* (See Remark 1.7); Finally, let $\text{MU}(\mu)$ be (μ^2) without the leading existential quantifier.

Theorem 2.2 *From a proof in RCA_0^A that $\text{UWKL}^+ \leftrightarrow \Pi_1^0\text{-TRANS}$, terms s, t can be extracted such that RCA_0^ω proves:*

$$(\forall\mu^2)[\text{MU}(\mu) \rightarrow \text{UWKL}(s(\mu))] \wedge (\forall\Phi^{1\rightarrow 1})[\text{UWKL}(\Phi) \rightarrow \text{MU}(t(\Phi, \Xi))]. \quad (2.1)$$

where Ξ is an extensionality functional for Φ .

Proof. Due to space constraints, we shall only prove that $\text{UWKL}^+ \rightarrow \Pi_1^0\text{-TRANS}$ in RCA_0^A , and obtain the associated second conjunct of (2.1), which is the most interesting result anyway. The remaining results are proved in Section A.2.

To prove $\text{UWKL}^+ \rightarrow \Pi_1^0\text{-TRANS}$ in RCA_0^A , assume UWKL^+ and suppose $\Pi_1^0\text{-TRANS}$ is false, i.e. there is f such that $(\forall^{\text{st}} n)f(n) = 0 \wedge (\exists m^0)f(m) \neq 0$. Now define the tree T_i for $i = 0, 1$ as follows

$$\sigma \in T_i \leftrightarrow [(\forall n^0 < |\sigma|)(\sigma(n) = i) \vee [(\forall n^0 < |\sigma|)(\sigma(n) = 1-i) \wedge (\forall m \leq |\sigma|)f(m) = 0]].$$

Note that $T_0 \approx_1 T_1$ but that the former (resp. the latter) only has one path, namely $00\dots$ (resp. $11\dots$). Hence, we must have $\Phi(T_0) \approx_1 \Phi(T_1)$ for Φ as in UWKL^+ , which however contradicts the standard extensionality of Φ . In light of this contradiction, we have $\text{UWKL}^+ \rightarrow \Pi_1^0\text{-TRANS}$.

Finally, we shall prove the second conjunct in (2.1). Note that $\Pi_1^0\text{-TRANS}$ can be brought into the following normal form: $(\forall^{\text{st}} f^1)(\exists^{\text{st}} i^0)[(\exists n^0)f(n) = 0 \rightarrow (\exists m \leq i)f(m) = 0]$, where the formula in square brackets is abbreviated by $B(f, i)$. Similarly, the second conjunct of UWKL^+ has the following normal form:

$$(\forall^{\text{st}} T^1, S^1, k^0)(\exists^{\text{st}} N)(\overline{TN} =_0 \overline{SN} \rightarrow \overline{\Phi(T)}k =_0 \overline{\Phi(S)}k), \quad (2.2)$$

which is immediate by resolving ‘ \approx_1 ’; We denote the formula in square brackets by $A(T, S, N, k, \Phi)$. Hence, $\text{UWKL}^+ \rightarrow \Pi_1^0\text{-TRANS}$ is easily seen to imply:

$$(\forall^{\text{st}} \Phi, \Xi)[[(\forall^{\text{st}} T^1)\text{UWKL}(\Phi, T) \wedge (\forall^{\text{st}} U^1, S^1, k^0)A(U, S, \Xi(U, S, k), k, \Phi)] \rightarrow (\forall^{\text{st}} f^1)(\exists^{\text{st}} n)B(f, n)]$$

Dropping the ‘st’ in the antecedent of the implication and brining out the remaining standard quantifiers, we obtain

$$(\forall^{\text{st}} \Phi, \Xi, f^1)(\exists^{\text{st}} n)[[(\forall T^1)\text{UWKL}(\Phi, T) \wedge (\forall U^1, S^1, k^0)A(U, S, \Xi(U, S, k), k, \Phi)] \rightarrow B(f, n)]$$

Let $C(\Phi, \Xi, f, n)$ be the formula in big square brackets and apply Corollary 1.4 to ‘ $\text{RCA}_0^A \vdash (\forall^{\text{st}} \Phi, \Xi, f^1)(\exists^{\text{st}} n)C(\Phi, \Xi, f, n)$ ’ to obtain a term t such that RCA_0^ω proves $(\forall \Phi, \Xi, f^1)(\exists n \in t(\Phi, \Xi, f))C(\Phi, \Xi, f, n)$. Now define the term $s(\Phi, \Xi, f)$ as $\max_{i < |t(\Phi, \Xi, f)|} t(\Phi, \Xi, f)(i)$ and note that s provides the functional (μ^2) if Φ satisfies $(\forall T^1)\text{UWKL}(\Phi, T)$ and Ξ is the associated extensionality functional. \square

Thus, we have obtained the effective equivalence $\text{UWKL} \leftrightarrow (\mu^2)$ from the associated nonstandard equivalence $\Pi_1^0\text{-TRANS} \leftrightarrow \text{UWKL}^+$. Now, Kohlenbach proves the equivalence $(\mu^2) \leftrightarrow \text{UWKL}$ in [5], and also established a related *effective* equivalence. Hence, the final equivalence in Theorem 2.2 is not that surprising, but our methodology arguably is: The *effective* equivalence in the latter theorem is obtained *automatically* (in the sense that the terms s, t can be ‘read off’ from the nonstandard proof in RCA_0^A) from a proof in which no attention to computational content is given, and Nonstandard Analysis is even used.

In the aforementioned proof from [5], Φ from UWKL is shown to be discontinuous, and *Grilliot’s trick* is then applied to obtain the Halting problem. Intuitively

speaking, the proof of Theorem 2.2 is similar: Φ from UWKL^+ is *nonstandard discontinuous* in light of $T_0 \approx_1 T_1 \wedge \Phi(T_0) \not\approx_1 \Phi(T_1)$, and the latter property allows us to derive a contradiction from the combination of standard extensionality and the negation of $\Pi_1^0\text{-TRANS}$. The Transfer principle $\Pi_1^0\text{-TRANS}$ becomes (μ^2) (which also solves the Halting problem) after applying Corollary 1.4.

2.2 Group theory and order

In this section, we prove a nonstandard equivalence for Levi's theorem for countable abelian groups from [3] and, as in the previous section, extract an effective equivalence. We also study the *contraposition* of Levi's theorem in the same way, yielding rather different results. Group theory is introduced in RM in [12, III.6].

Definition 2.3 ([3, §2]) *Let A be a countable abelian group. A is torsion-free if $(\forall n^0)(\forall a \in A \setminus \{0_A\})(n \times a \neq 0)$. A is orderable if there exists a linear ordering ' $<$ ' on A such that $(\forall a, b, c \in A)(a \leq b \rightarrow a + c \leq b + c)$. P is the positive cone of A if $P = \{a \in A : a \geq 0\}$.*

Let ORD be Levi's theorem that every torsion-free countable abelian group is orderable. Define $\text{UORD}(\Phi, A)$ as the statement that $\Phi(A)$ is the order for A as in ORD. Finally, define UORD^+ as follows:

$$(\exists^{\text{st}} \Phi^{1 \rightarrow 1})[(\forall^{\text{st}} A^1) \text{UORD}(\Phi, A) \wedge (\forall^{\text{st}} A^1, B^1)(A \approx_1 B \rightarrow \Phi(A) \approx_1 \Phi(B))].$$

We have the following theorem.

Theorem 2.4 *From a proof in RCA_0^A that $\text{UORD}^+ \leftrightarrow \Pi_1^0\text{-TRANS}$, terms s, t can be extracted such that RCA_0^ω proves:*

$$(\forall \mu^2)[\text{MU}(\mu) \rightarrow \text{UORD}(s(\mu))] \wedge (\forall \Phi^{1 \rightarrow 1})[\text{UORD}(\Phi) \rightarrow \text{MU}(t(\Phi, \Xi))]. \quad (2.3)$$

where Ξ is an extensionality functional for Φ .

Proof. Given the similarity in syntactic structure between ORD and WKL, the proof of $\Pi_1^0\text{-TRANS} \rightarrow \text{UORD}^+$ in RCA_0^A follows immediately from the proof of $\text{WKL} \rightarrow \text{ORD}$ ([3, Lemma 2.3]) and the proof of Theorem 2.2 in Section A.2. The proof that $\text{UORD}^+ \rightarrow \Pi_1^0\text{-TRANS}$ proceeds as follows. Note that the ordering provided by ORD is not unique: If ' \leq ' is an order on A , then ' \sqsubseteq ' obtained by $a \sqsubseteq b \equiv \neg(a \leq b)$ is also one, and the associated positive cones only intersect in the neutral element 0_A . Now assume UORD^+ and suppose that $\Pi_1^0\text{-TRANS}$ is false, i.e. there is standard f^1 such that $(\forall^{\text{st}} n^0)f(n) = 0$ and $(\exists m^0)f(m) \neq 0$. Take a standard torsion-free abelian group $A^1 = (a_0, a_2, a_3, \dots)$, and define the following standard group:

$$B(i) := \begin{cases} -A(i) & (\exists n \leq \max\{A(i), -A(i)\})f(n) \neq 0 \\ A(i) & \text{otherwise,} \end{cases} \quad (2.4)$$

Note that the ‘inverse’ operation ‘ $-$ ’ associated to A is also standard, implying that $-a$ is standard if and only if $a \in A$ is. As a result, the modification via f to A in (2.4) does not change the standard part of A , i.e. we have $A \approx_1 B$. However, we also have $\Phi(A) \not\approx_1 \Phi(B)$, as (2.4) switches $A(i)$ and $-A(i)$ in the enumeration of A for large enough i . In particular, the positive cone of A only intersects the positive cone of B in $0_A = 0_B$ due to the ‘switch’ taking place in the first clause of (2.4). The previous contradiction yields $\text{UORD}^+ \rightarrow \Pi_1^0\text{-TRANS}$.

Finally, in light of the similarity in syntactic structure between UORD^+ and UWKL^+ , one obtains (2.3) from $\text{UORD}^+ \leftrightarrow \Pi_1^0\text{-TRANS}$ in the same way as one obtains (2.1) from $\text{UWKL}^+ \leftrightarrow \Pi_1^0\text{-TRANS}$, and we are done. \square

Thus, (2.3) establishes the effective equivalence between WKL and ORD , and we now study the effective equivalence between the *contrapositions* of the latter, leading to *quite* different results. Recall that the fan theorem, denoted FAN , is the classical contraposition of weak König’s lemma. Similarly, let DRO be the contraposition of ORD and consider the following explicit versions:

$$\begin{aligned} (\forall T^1 \leq_1 1, g^2)[(\forall \beta \leq_1 1)\bar{\beta}g(\beta) \notin T & \quad (\text{UFAN}(\Phi)) \\ \rightarrow (\forall \beta \leq_1 1)(\exists i \leq \Phi(g))\bar{\beta}i \notin T]. \end{aligned}$$

$$\begin{aligned} (\forall A^1, h^2)[(\forall X^1)(\exists a, b, c \in A)(a, b, c \leq h(X) \wedge a + c >_X b + c) & \quad (\text{UDRO}(\Psi)) \\ \rightarrow (\exists n^0, a \in A)(n, a \leq \Psi(h) \wedge n \times a = 0_A)]. \end{aligned}$$

We have the following theorem.

Theorem 2.5 *From the proof of $\text{WKL} \leftrightarrow \text{ORD}$ in RCA_0 (See [3, p. 179]), terms s, t can be extracted witnessing the explicit equivalence $\text{FAN} \leftrightarrow \text{ORD}$ in RCA_0^ω :*

$$(\forall \Phi^3)[\text{UFAN}(\Phi) \rightarrow \text{UDRO}(s(\Phi))] \wedge (\forall \Psi^3)[\text{UDRO}(\Psi) \rightarrow \text{UFAN}(t(\Psi))]. \quad (2.5)$$

Proof. First of all, assume the following two theorems are equivalent in RCA_0^A .

$$\begin{aligned} (\forall^{\text{st}} T^1 \leq_1 1, g^2)[(\forall \beta \leq_1 1)\bar{\beta}g(\beta) \notin T & \quad (2.6) \\ \rightarrow (\exists^{\text{st}} k)(\forall \beta \leq_1 1)(\exists i \leq k)\bar{\beta}i \notin T]. \end{aligned}$$

$$\begin{aligned} (\forall^{\text{st}} A^1, h^2)[(\forall X^1)(\exists a, b, c \in A)(a, b, c \leq h(X) \wedge a + c >_X b + c) & \quad (2.7) \\ \rightarrow (\exists^{\text{st}} m)(\exists n^0, a \in A)(n, a \leq m \wedge n \times a = 0_A)]. \end{aligned}$$

Applying Corollary 1.4 to the normal forms of ‘(2.6) \rightarrow (2.7)’ and ‘(2.7) \rightarrow (2.6)’ now immediately yields (2.5). It is a straightforward verification that the proof that $\text{ORD} \leftrightarrow \text{WKL}$ in [3, Theorem 2.5] can be (easily) modified to a proof of ‘(2.6) \leftrightarrow (2.7)’ in RCA_0^A . For completeness, we prove one direction of the latter implication in Section A.3. \square

The principle $(\exists\Phi)\text{UFAN}(\Phi)$ is conservative over weak König's lemma by [4, Prop. 3.15], while (μ^2) is essentially arithmetical comprehension. Hence, there is a big difference in strength between UORD and $(\exists\Psi)\text{UDRO}(\Psi)$, which can be explained by the different constructive status of ORD and DRO (See [4, p. 294]).

Finally, we show that from certain effective implications, one can re-obtain the original nonstandard theorem. To this end, consider:

$$(\forall\Phi, \Xi, f)[\text{UWKL}_{\text{pw}}(\Phi, i) \wedge \text{EXT}_{\text{pw}}(\Phi, \Xi, i) \rightarrow \text{MU}_{\text{pw}}(o(\Phi, \Xi, f), f) \quad (\text{HER}(i, o))$$

where $\text{MU}_{\text{pw}}(\mu, f)$ is $\text{MU}(\mu)$ with the leading universal quantifier dropped, where $\text{UWKL}_{\text{pw}}(\Phi, i)$ is $(\forall T^1 \in i(\Phi, \Xi, f)(1))\text{UWKL}(\Phi, T)$, and where $\text{EXT}_{\text{pw}}(\Phi, \Xi, i)$ is

$$(\forall U, S, k \in i(\Phi, \Xi, f)(2))(\overline{S}\Xi(U, S, k) = \overline{U}\Xi(U, S, k) \rightarrow \overline{\Phi(U)}k = \overline{\Phi(S)}k).$$

Intuitively speaking $\text{HER}(i, o)$ expresses that in order to compute Feferman's mu-operator via o for one single f^1 , one needs to supply Φ which satisfies UWKL but only for the finite sequence of trees given by i , and similar for Ξ . In other words, $\text{HER}(i, o)$ is a 'pointwise' version of the second conjunct of (2.1).

Theorem 2.6 *From $\text{RCA}_0^A \vdash \text{UWKL}^+ \rightarrow \Pi_1^0\text{-TRANS}$, closed terms i, o can be extracted such that $\text{RCA}_0^\omega \vdash \text{HER}(i, o)$. If for closed terms i, o , RCA_0^ω proves $\text{HER}(i, o)$, then the latter proof yields a proof of $\text{UWKL}^+ \rightarrow \Pi_1^0\text{-TRANS}$ in RCA_0^A .*

Proof. For the first part of the theorem, consider the proof of Theorem 2.2, in particular the sentence below (2.2) as follows:

$$(\forall^{\text{st}}\Phi, \Xi)[[(\forall^{\text{st}}T^1)\text{UWKL}(\Phi, T) \wedge (\forall^{\text{st}}U^1, S^1, k^0)A(U, S, \Xi(U, S, k), k)] \rightarrow (\forall^{\text{st}}f^1)(\exists^{\text{st}}n)B(f, n)].$$

Bringing the previous into normal form, *without* dropping 'st' predicates, yields

$$(\forall^{\text{st}}\Phi, \Xi, f^1)(\exists^{\text{st}}T, U, S, k, n)[[\text{UWKL}(\Phi, T) \wedge A(U, S, \Xi(U, S, k), k)] \rightarrow B(f, n)],$$

where $D(\Phi, \Xi, f, T, U, S, k, n)$ abbreviates the formula in big square brackets. Applying Corollary 1.4 to the fact that RCA_0^A proves

$$(\forall^{\text{st}}\Phi, \Xi, f^1)(\exists^{\text{st}}T, U, S, k, n)D(\Phi, \Xi, f, T, U, S, k, n)$$

yields a term t such that RCA_0^ω proves:

$$(\forall\Phi, \Xi, f^1)(\exists T, U, S, k, n \in t(\Phi, \Xi, f))D(\Phi, \Xi, f, T, U, S, k, n). \quad (2.8)$$

Now define the term o as the maximum of all entries of t pertaining to n , and define $i(\Phi, \Xi, f)(1)$ as the sequence of all entries of t pertaining to T , and the same for $i(\Phi, \Xi, f)(2)$ and U, S, k . With these definitions, (2.8) implies $\text{HER}(i, o)$.

For the second part of the proof, by the second standardness axiom from Definition 1.1, terms like i, o are standard in RCA_0^A . Hence, if $\text{RCA}_0^\omega \vdash \text{HER}(i, o)$, then RCA_0^A proves $\text{HER}(i, o) \wedge \text{st}(i) \wedge \text{st}(o)$. Thus, for standard Φ, Ξ, f , $i(\Phi, \Xi, f)$

and $o(\Phi, \Xi, f)$ are standard, yielding that $\text{MU}_{\text{pw}}(o(\Phi, \Xi, f), f)$ for standard inputs implies $(\exists n)f(n) = 0 \rightarrow (\exists^{\text{st}} m)f(m) = 0$. Hence, $\text{HER}(i, o)$ yields

$$(\forall^{\text{st}} \Phi, \Xi, f) [\text{UWKL}_{\text{pw}}(\Phi, i) \wedge \text{EXT}_{\text{pw}}(\Phi, \Xi, i) \rightarrow [(\exists n)f(n) = 0 \rightarrow (\exists^{\text{st}} m)f(m) = 0]],$$

and bring two of the standard universal quantifiers inside as follows:

$$(\forall^{\text{st}} f) [(\exists^{\text{st}} \Phi, \Xi) [\text{UWKL}_{\text{pw}}(\Phi, i) \wedge \text{EXT}_{\text{pw}}(\Phi, \Xi, i)] \rightarrow [(\exists n)f(n) = 0 \rightarrow (\exists^{\text{st}} m)f(m) = 0]],$$

and note that UWKL^+ implies the antecedent of the previous, i.e.

$$(\forall^{\text{st}} f) [\text{UWKL}^+ \rightarrow [(\exists n)f(n) = 0 \rightarrow (\exists^{\text{st}} m)f(m) = 0]],$$

which immediately yields $\text{UWKL}^+ \rightarrow \Pi_1^0\text{-TRANS}$. \square

We refer to $\text{HER}(i, o)$ as the *Herbrandisation* of the implication $\text{UWKL}^+ \rightarrow \Pi_1^0\text{-TRANS}$. As suggested by the notation, the Herbrandisation is a ‘pointwise’ version of the second conjunct of (2.1). We could obtain Hebrandisations of e.g. (2.3), but do not go into details due to space constraints.

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A Technical Appendix

A.1 Proof of Corollary 1.3

In this section, we prove Corollary 1.3, as follows.

Corollary A.1 *If for internal ψ , $\Phi(\underline{a}) \equiv (\forall^{\text{st}} \underline{x})(\exists^{\text{st}} \underline{y})\psi(\underline{x}, \underline{y}, \underline{a})$ satisfies (1.1), then $(\forall \underline{x})(\exists \underline{y} \in t(\underline{x}))\psi(\underline{x}, \underline{y}, \underline{a})$ is proved in the corresponding formula (1.2).*

Proof. Clearly, if for ψ and Φ as given we have $\Phi(\underline{a})^{S_{\text{st}}} \equiv \Phi(\underline{a})$, then the corollary follows immediately from the theorem. A tedious but straightforward verification using the clauses (i)-(v) in [2, Def. 7.1] establishes that indeed $\Phi(\underline{a})^{S_{\text{st}}} \equiv \Phi(\underline{a})$. For completeness, we now list these five inductive clauses and perform this verification.

Hence, if $\Phi(\underline{a})$ and $\Psi(\underline{b})$ in the language of \mathbf{P} have the following interpretations

$$\Phi(\underline{a})^{S_{\text{st}}} \equiv (\forall^{\text{st}} \underline{x})(\exists^{\text{st}} \underline{y})\varphi(\underline{x}, \underline{y}, \underline{a}) \text{ and } \Psi(\underline{b})^{S_{\text{st}}} \equiv (\forall^{\text{st}} \underline{u})(\exists^{\text{st}} \underline{v})\psi(\underline{u}, \underline{v}, \underline{b}), \quad (\text{A.1})$$

then they interact as follows with the logical connectives by [2, Def. 7.1]:

- (i) $\psi^{S_{\text{st}}} := \psi$ for atomic internal ψ .
- (ii) $(\text{st}(z))^{S_{\text{st}}} := (\exists^{\text{st}} x)(z = x)$.
- (iii) $(\neg\Phi)^{S_{\text{st}}} := (\forall^{\text{st}} \underline{Y})(\exists^{\text{st}} \underline{x})(\forall \underline{y} \in \underline{Y}[\underline{x}])\neg\varphi(\underline{x}, \underline{y}, \underline{a})$.
- (iv) $(\Phi \vee \Psi)^{S_{\text{st}}} := (\forall^{\text{st}} \underline{x}, \underline{u})(\exists^{\text{st}} \underline{y}, \underline{v})[\varphi(\underline{x}, \underline{y}, \underline{a}) \vee \psi(\underline{u}, \underline{v}, \underline{b})]$
- (v) $((\forall z)\Phi)^{S_{\text{st}}} := (\forall^{\text{st}} \underline{x})(\exists^{\text{st}} \underline{y})(\forall z)(\exists \underline{y}' \in \underline{y})\varphi(\underline{x}, \underline{y}', z)$

Hence, fix $\Phi_0(\underline{a}) \equiv (\forall^{\text{st}} \underline{x})(\exists^{\text{st}} \underline{y})\psi_0(\underline{x}, \underline{y}, \underline{a})$ with internal ψ_0 , and note that $\phi^{S_{\text{st}}} \equiv \phi$ for any internal formula. We have $[\text{st}(\underline{y})]^{S_{\text{st}}} \equiv (\exists^{\text{st}} \underline{w})(\underline{w} = \underline{y})$ and also

$$[\neg \text{st}(\underline{y})]^{S_{\text{st}}} \equiv (\forall^{\text{st}} \underline{W})(\exists^{\text{st}} \underline{x})(\forall \underline{w} \in \underline{W}[\underline{x}])\neg(\underline{w} = \underline{y}) \equiv (\forall^{\text{st}} \underline{w})(\underline{w} \neq \underline{y}).$$

Hence, $[\neg \text{st}(\underline{y}) \vee \neg \psi_0(\underline{x}, \underline{y}, \underline{a})]^{S_{\text{st}}}$ is just $(\forall^{\text{st}} \underline{w})[(\underline{w} \neq \underline{y}) \vee \neg \psi_0(\underline{x}, \underline{y}, \underline{a})]$, and

$$[(\forall \underline{y})[\neg \text{st}(\underline{y}) \vee \neg \psi_0(\underline{x}, \underline{y}, \underline{a})]]^{S_{\text{st}}} \equiv (\forall^{\text{st}} \underline{w})(\exists^{\text{st}} \underline{v})(\forall \underline{y})(\exists \underline{v}' \in \underline{v})[\underline{w} \neq \underline{y} \vee \neg \psi_0(\underline{x}, \underline{y}, \underline{a})].$$

which is just $(\forall^{\text{st}} \underline{w})(\forall \underline{y})[(\underline{w} \neq \underline{y}) \vee \neg \psi_0(\underline{x}, \underline{y}, \underline{a})]$. Furthermore, we have

$$\begin{aligned} [(\exists^{\text{st}} \underline{y})\psi_0(\underline{x}, \underline{y}, \underline{a})]^{S_{\text{st}}} &\equiv [\neg(\forall \underline{y})[\neg \text{st}(\underline{y}) \vee \neg \psi_0(\underline{x}, \underline{y}, \underline{a})]]^{S_{\text{st}}} \\ &\equiv (\forall^{\text{st}} \underline{V})(\exists^{\text{st}} \underline{w})(\forall \underline{v} \in \underline{V}[\underline{w}])\neg[(\forall \underline{y})[(\underline{w} \neq \underline{y}) \vee \neg \psi_0(\underline{x}, \underline{y}, \underline{a})]] \\ &\equiv (\exists^{\text{st}} \underline{w})(\exists \underline{y})[(\underline{w} = \underline{y}) \wedge \psi_0(\underline{x}, \underline{y}, \underline{a})] \equiv (\exists^{\text{st}} \underline{w})\psi_0(\underline{x}, \underline{w}, \underline{a}). \end{aligned}$$

Hence, we have proved so far that $(\exists^{\text{st}} \underline{y})\psi_0(\underline{x}, \underline{y}, \underline{a})$ is invariant under S_{st} . By the previous, we also obtain:

$$[\neg \text{st}(\underline{x}) \vee (\exists^{\text{st}} \underline{y})\psi_0(\underline{x}, \underline{y}, \underline{a})]^{S_{\text{st}}} \equiv (\forall^{\text{st}} \underline{w}')[(\exists^{\text{st}} \underline{w})[(\underline{w}' \neq \underline{x}) \vee \psi_0(\underline{x}, \underline{w}, \underline{a})]].$$

Our final computation now yields the desired result:

$$\begin{aligned}
[(\forall^{\text{st}} \underline{x})(\exists^{\text{st}} y)\psi_0(\underline{x}, \underline{y}, \underline{a})]^{S_{\text{st}}} &\equiv [(\forall \underline{x})(\neg \text{st}(\underline{x}) \vee (\exists^{\text{st}} y)\psi_0(\underline{x}, \underline{y}, \underline{a}))]^{S_{\text{st}}} \\
&\equiv (\forall^{\text{st}} \underline{w}')(\exists^{\text{st}} \underline{w})(\forall \underline{x})(\exists \underline{w}'' \in \underline{w})[(\underline{w}' \neq \underline{x}) \vee \psi_0(\underline{x}, \underline{w}'', \underline{a})]. \\
&\equiv (\forall^{\text{st}} \underline{w}')(\exists^{\text{st}} \underline{w})(\exists \underline{w}'' \in \underline{w})\psi_0(\underline{w}', \underline{w}'', \underline{a}).
\end{aligned}$$

The last step is obtained by taking $\underline{x} = \underline{w}'$. Hence, we may conclude that the normal form $(\forall^{\text{st}} \underline{x})(\exists^{\text{st}} y)\psi_0(\underline{x}, \underline{y}, \underline{a})$ is invariant under S_{st} , and we are done. \square

Note that the previous proof may also be found in [7–9].

A.2 Full proof of Theorem 2.2

In this section, we establish the part of Theorem 2.5 not covered by the proof in Section 2.1.

Proof. We first prove $\Pi_1^0\text{-TRANS} \rightarrow \text{UWKL}^+$ in RCA_0^A . To this end, consider the functional ν^2 defined as follows:

$$\nu(f^1, M^0) := \begin{cases} (\mu n \leq M)f(n) = 0 & \text{if such exists} \\ 0 & \text{otherwise} \end{cases}.$$

Thanks to $\Pi_1^0\text{-TRANS}$, we have $(\forall^{\text{st}} f^1)(\forall M, N \in \Omega)(\nu(f, M) =_0 \nu(f, N))$, and applying underspill (which is available in RCA_0^A due to [2, §5.3]), we obtain

$$(\forall^{\text{st}} f^1)(\exists^{\text{st}} n^0)(\forall M, N \geq n)(\nu(f, M) =_0 \nu(f, N)), \quad (\text{A.2})$$

and applying HAC_{int} to (A.2) yields $\Psi^{1 \rightarrow 0^*}$ and $\Phi(f) := \max_{i < |\Psi(f)|} \Psi(f)(i)$, such that $\nu(\cdot, \Phi(\cdot))$ is essentially Feferman's operator relative to 'st', i.e. we have shown $(\mu^2)^{\text{st}}$. However, (μ^2) implies arithmetical comprehension as follows:

$$(\forall f^0)[(\exists n^0)f(n) = 0 \leftrightarrow f(\mu(f)) = 0],$$

which in turn yields weak König's lemma (See [12, §2]). By the previous, we have WKL^{st} , and applying $\Pi_1^0\text{-TRANS}$ to the consequent, we obtain that

$$(\forall^{\text{st}} T^1 \leq_1 1)[(\forall n)(\exists \alpha)(|\alpha| = n \wedge \alpha \in T) \rightarrow (\exists^{\text{st}} \beta^1 \leq_1 1)(\forall m)(\overline{\beta}m \in T)] \quad (\text{A.3})$$

Now bring outside the standard quantifier ' $(\exists^{\text{st}} \beta^1 \leq_1 1)$ ' and apply HAC_{int} to the resulting formula to obtain $\Psi^{1 \rightarrow 1^*}$. By definition, if T is a standard infinite binary tree, one of the entries of $\Psi(T)$ is a (standard) path through T . Thanks to $(\mu^2)^{\text{st}}$ and $\Pi_1^0\text{-TRANS}$, we can test *which* entry of Ψ is such a path, and define $\Phi^{1 \rightarrow 1}$ to be the $\Psi(T)(i)$ with that property, where i is least. Hence, we obtain the first conjunct of UWKL^+ . The second conjunct of the latter now easily follows by applying $\Pi_1^0\text{-TRANS}$ to the axiom of extensionality (E) pertaining to Φ .

By the previous, RCA_0^A proves $\Pi_1^0\text{-TRANS} \rightarrow \text{UWKL}^+$, and the latter implies, in the same way as in the proof of Theorem 2.2, that

$$(\forall^{\text{st}} \mu^2)[\text{MU}(\mu) \rightarrow (\text{A.3})].$$

Bringing the previous into normal form and applying Corollary 1.4, one obtains a term t for which one entry of $t(\mu, T)$ is a path through T , if T is infinite and μ is as in (μ^2) . It is trivial to use (μ^2) to find the correct entry of $t(\mu, T)$. \square

A.3 Full proof of Theorem 2.5

In this section, we establish the part of Theorem 2.2 not covered by the proof in Section 2.2.

Proof. The proof of Theorem 2.5 hinges on the equivalence (2.6) \leftrightarrow (2.7), the forward direction of which we establish now, based on [3, Lemma 2.3].

First of all, the proof of $\text{WKL} \rightarrow \text{ORD}$ in RCA_0 in the aforementioned lemma proceeds by defining for a countable torsion-free group A , a certain binary tree T^A and associated set P_σ^A . This tree T^A is proved to be infinite, and the path provided by WKL (this being the only use of WKL) is used to define the positive cone P^A of A (See Definition 2.3) via $P^A := \cup_\sigma P_\sigma^A$, where each σ is an initial segment of the aforementioned path.

Secondly, in light of the previous paragraph, the binary tree T^A and set P_σ^A may be defined in RCA_0^A . By the definition of these objects in [3, p. 178], they are standard. Furthermore, since all recursor constants are standard (by Definition 1.1), we may use Π_1^0 -induction and its relativisation to ‘st’. Hence, RCA_0^A also proves that the tree T is infinite relative to ‘st’ (in the sense that it includes binary type zero sequences of any standard length).

Thirdly, working in $\text{RCA}_0^A + (2.6)$ let A be a standard countable abelian group such that $(\forall^{\text{st}} n, a \in A)(n \times a \neq 0_A)$, i.e. A is torsion-free relative to ‘st’. By the contraposition of (2.6), the tree T^A is such that $(\forall^{\text{st}} g^2)(\exists \beta \leq_1 1) \bar{\beta}g(\beta) \in T^A$. For standard h^2 , let $\beta^1 \leq_1 1$ be such a sequence and define $P^{A,h} := \cup_{\sigma \preceq \bar{\beta}3h(\beta)+3} P_\sigma$. Clearly, $P^{A,h}$ is a ‘partial’ positive cone of A , and we have $(\exists X^1)(\forall a, b, c \in A)(a, b, c \leq h(X) \rightarrow a + c \leq_X b + c)$. Hence, we have proved that for standard A^1, h^2 , we have the contraposition of (2.7). In short, $\text{RCA}_0^A \vdash (2.6) \rightarrow (2.7)$.

Finally, the reverse implication can be proved in exactly the same way based on the proof of [3, Theorem 2.5]. However, this involves proving a version of (2.6) \leftrightarrow (2.7) for **FAN** and [12, IV.4.4.3], where the latter is the statement that the ranges of two non-overlapping functions can be separated by a set. Although this is quite straightforward, it is beyond the scope of this appendix. \square